FUNDAMENTAL SOLUTIONS IN PLANE PROBLEM FOR ANISOTROPIC ELASTIC MEDIUM UNDER MOVING OSCILLATING SOURCE

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Abstract

In present article we consider the problems of concentrated point force which is moving with constant velocity and oscillating with cyclic frequency in unbounded homogeneous anisotropic elastic two-dimensional medium. The properties of plane waves and their phase, slowness and ray or group velocity curves for 2D problem in moving coordinate system are described. By using the Fourier integral transform techniques and established the properties of the plane waves, the explicit representation of the elastodynamic Green's tensor is obtained for all types of source motion as a sum of the integrals over the finite interval. The dynamic components of the Green's tensor are extracted.

The stationary phase method is applied to derive an asymptotic approximation of the far wave field. The simple formulae for Poynting energy flux vectors for moving and fixed observers are presented too. It is noted that in the far zones the cylindrical waves are separated under kinematics and energy.

It is shown that the motion bring some differences in the far field properties. They are modification of the wave propagation zones and their number, fast and slow waves appearance under trans- and superseismic motion and so on.

1 Introduction

The theory of elasticity with moving source have a range of important applications. They are linked with the developing of more high-speed transport and with the necessity to evaluate the influence of elastic waves from moving objects on different constructions. The theoretical side of these problems is also of interest. The changes in the character of mechanical fields is essentially depend on the behavior of source motion and are caused by the type of differential equations ranging from elliptical to hyperbolic.

From the steady-state problems with moving sources the problems with moving and oscillating sources are the most complicated. We use the following terminology.

If the source is moving with constant velocity $\mathbf{w}$ and simultaneously oscillating with frequency ω, we consider the problem B, while in the case $\mathbf{w} = 0$, we have the classical harmonic problems with oscillating source and so refer them to the problem A.

Nowadays considerable progress is obtained in the investigation of the problem B. The correspondence principles between the problems B and A [1], analogous known in fluid mechanic [2], are stated. The principles for unique solution selection are investigated [1, 3]. The energetic principles are formulated and the general theorems about energy transport are established [3], the some actual problems B for isotropic media are solved [4–6].

It is evident, that for anisotropic elastic media the particular problems B are not studied sufficiently. In [7] the plane waves and fundamental solutions for problem B for tree-dimensional anisotropic media are studied, and in [8] the approaches for the problem B for anisotropic elastic and piezoelectric waveguides are proposed.

In present paper we investigate the properties of plane waves and fundamental solutions in the problem B for anisotropic elastic plane.

It is well-known, that for many classes of anisotropic materials it is possible to formulate the problems A and B in conditions of plane deformation. For existence of the plane problem several elastic modules must be equal to zero. Thus, for the plane deformation in the plane $0x_1x_3$ for elastic modules in two-index notation we require that $C_{14} = C_{34} = C_{45} = C_{16} = C_{36} = C_{56} = 0$, for the plane deformation in the plane $0x_1x_2$ $C_{14} = C_{24} = C_{46} = C_{15} = C_{25} = C_{56} = 0$, and for the plane deformation in the plane $0x_2x_3$ $C_{25} = C_{26} = C_{35} = C_{36} = C_{45} = C_{46} = 0$.

The vectors of external force in the plane deformation problems must not have non-zero components in the perpendicular direction and must not depend on perpendicular to plane coordinate.

We assume that all above-mentioned conditions are implemented. To be more concrete, let us accept that the plane deformation is realized in plane $0x_1x_2$, and therefore the vector of mechanic displacement has non-zero components u_1 and u_2: $\mathbf{u} = \{u_1(\mathbf{x}, t), u_2(\mathbf{x}, t)\}$, $\mathbf{x} = \{x_1, x_2\}$. For harmonic vibrations with frequency ω we shall find the established solution in the form

$$\mathbf{u} = \mathbf{v}\exp(i\omega t). \tag{1.1}$$

In the problem with moving sources we consider two coordinate systems. Let $\{\xi_1, \xi_2\}$ be fixed coordinate system with time τ, and let $\{x_1, x_2\}$ be coordinate system, which is moving relatively to fixed system with constant velocity $\mathbf{w} = \{w_1, w_2\}$. We denote t the time in moving coordinate system. The two coordinate systems are connected with each other by the following relations

$$x_1 = \xi_1 - w_1\tau, \quad x_2 = \xi_2 - w_2\tau, \quad t = \tau. \tag{1.2}$$

For the problem B we propose that in moving coordinate system the harmonic behavior (1.1) with frequency ω is existed.

For the equations of theory of elasticity in moving coordinate system we use the following relations from (1.2)

$$\nabla^\xi = \nabla^x, \quad \partial_\tau = \partial_t - \mathbf{w} \cdot \nabla^x, \tag{1.3}$$

where $\nabla^x = \{\partial/\partial x_1, \partial/\partial x_2\} = \{\partial_1, \partial_2\}$, $\nabla^\xi = \{\partial/\partial\xi_1, \partial/\partial\xi_2\}$.

Then the stain vector $\mathbf{S} = \{\varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12}\}$ (ε_{ij} are the components of strain tensor) may be expressed from the displacement vector as

$$\mathbf{S} = \mathbf{L}(\nabla^x) \cdot \mathbf{u}, \tag{1.4}$$

where

$$\mathbf{L}(\nabla^x) = \begin{pmatrix} \partial_1 & 0 \\ 0 & \partial_2 \\ \partial_2 & \partial_1 \end{pmatrix}. \tag{1.5}$$

From Hook's low for anisotropic media for the plane deformation in the plane (x_1x_2) we have the relation between the stress vector $\mathbf{T} = \{\sigma_{11}, \sigma_{22}, \sigma_{12}\}$ (σ_{ij} are the components of stress tensor) and the strain vector $\mathbf{S}$

$$\mathbf{T} = \mathbf{C} \cdot \mathbf{S}, \tag{1.6}$$

where

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & C_{16} \\ & C_{22} & C_{26} \\ \mathrm{sym} & & C_{66} \end{pmatrix}. \tag{1.7}$$

We suppose here C_{ij} is not depended on coordinate, i.e. the medium material is homogeneous.

By using (1.3) and the previous notations the equations for elastic medium in the case of plane deformation may be written in the form

$$\mathbf{L}^*(\nabla^x) \cdot \mathbf{T} + \mathbf{f} = \rho(\partial_t - \mathbf{w} \cdot \nabla^x)^2 \mathbf{u}, \tag{1.8}$$

where ρ is the density ($\rho = \text{const}$), $\mathbf{f}$ is the body force, $(...)^*$ is the conjugation operator.

By using (1.4)—(1.6) the equation (1.8) may be rewritten

$$\mathbf{L}^*(\nabla^x) \cdot \mathbf{C} \cdot \mathbf{L}(\nabla^x) \cdot \mathbf{u} + \mathbf{f} = \rho(\partial_t - \mathbf{w} \cdot \nabla^x)^2 \mathbf{u}. \tag{1.9}$$

This equation together with the (1.1) is the equation for amplitude $\mathbf{v}$.

We study the properties of plane wave in problem B at first, to determine the fundamental solution.

2 Plane waves and their characteristic curves

We find the solution of equations (1.9) without body force ($\mathbf{f} = 0$) in the form of plane waves

$$\mathbf{u} = A\mathbf{p} \exp\left[i(\omega t - \boldsymbol{\alpha} \cdot \mathbf{x})\right], \tag{2.1}$$

where A is the amplitude, $\mathbf{p}$ is the unit polarization vector ($|\mathbf{p}| = 1$), $\boldsymbol{\alpha}$ is the wave vector ($\boldsymbol{\alpha} = \alpha\mathbf{n}$, $|\mathbf{n}| = 1$, $\mathbf{n}$ is the unit wave normal vector).

By substituting (2.1) into (1.9) yields the eigenvalue problem

$$\boldsymbol{\Gamma}(\mathbf{n}) \cdot \mathbf{p} = \rho\nu^2(\mathbf{n})\mathbf{p}, \tag{2.2}$$

where

$$\nu^2(\mathbf{n}) = (c_p^B(\mathbf{n}) + w_n)^2, \tag{2.3}$$

$w_n = \mathbf{w} \cdot \mathbf{n}$, $c_p^B(\mathbf{n}) = \omega/\alpha$ is the phase velocity in the problem B, $\mathbf{\Gamma}(\mathbf{n}) = \mathbf{L}^*(\mathbf{n}) \cdot \mathbf{C} \cdot \mathbf{L}(\mathbf{n})$ is the acoustic Christoffel's tensor (matrix).

As it follows from common properties of the problem A in 3D [9, 10], here in the problem A in 2D two plane waves with phase velocities $\nu_j(\mathbf{n}) = c_{pj}^A(\mathbf{n})$, $c_{pj}^A(\mathbf{n}) > 0$, $j = 1, 2$, and with polarization vector (eigenvector) $\mathbf{p}_j$ exist for arbitrary direction $\mathbf{n}$,which can be selected orthonormalized. For the problem B, as it is obvious from (2.2), the situation is more complicated. Particularly, for the direct plane waves ($c_{pj}^B > 0$) in the problem B with fixed $\mathbf{n}$ and $j \in \{1, 2\}$

$$(c_{pj}^A(\mathbf{n}) > w_n) \ \wedge \ (c_{pj}^A(\mathbf{n}) \geq -w_n) \quad \Rightarrow \quad c_{pj}^B(\mathbf{n}) = c_{pj}^A(\mathbf{n}) - w_n, \tag{2.4}$$

$$c_{pj}^A(\mathbf{n}) < -w_n \quad \Rightarrow \quad c_{pj}^{Bk}(\mathbf{n}) = (-1)^k c_{pj}^A(\mathbf{n}) - w_n; \quad k = 0, 1, \tag{2.5}$$

$$c_{pj}^A(\mathbf{n}) \leq w_n \quad \Rightarrow \quad \text{there are not direct plane waves.} \tag{2.6}$$

Thus, in condition (2.4) for fixed $\mathbf{n}$ and j we have one plane wave, in condition (2.5) we have two (one – $k = 0$ – fast, other – $k = 1$ — slow), and in condition (2.6) the direct plane waves are absent. The polarization vectors $\mathbf{p}_j$ of plane waves in the problems A and B are identical, with the exception of the case (2.6), besides in the case (2.5) both fast and slow waves have the same polarization vector. Henceforth we denote the phase velocity in the cases (2.4), (2.5) by uniform way: $c_{pj}^{B(k)} = c_{pj}^{B(k)}(\mathbf{n}) = (-1)^k c_{pj}^A(\mathbf{n}) - w_n$, $(k = 0) \vee (k = 0, 1)$, i.e. for the case (2.4) $c_{pj}^B = c_{pj}^{B(0)}$.

If for $\forall \mathbf{n}$, $\forall j \in \{1, 2\}$ the condition (2.4) is realized, then we shall call subseismic the motion behavior, and otherwise – trans- or superseismic. The difference between that notions we shall mark below.

We introduce the phase velocity vector $\mathbf{c}_{pj}^{B(k)}$ and the inverse velocity vector $\mathbf{L}_j^{B(k)}$

$$\mathbf{c}_{pj}^{B(k)} = c_{pj}^{B(k)} \mathbf{n} = (-1)^k \mathbf{c}_{pj}^A(\mathbf{n}) - w_n \mathbf{n}; \quad \mathbf{c}_{pj}^A(\mathbf{n}) = c_{pj}^A(\mathbf{n})\mathbf{n}, \tag{2.7}$$

$$\mathbf{L}_j^{B(k)} = \mathbf{n}/c_{pj}^{B(k)}. \tag{2.8}$$

By analogy with problem A, we shall also call the vector $\mathbf{L}_j^{B(k)}$ refraction vector or slowness vector.

The dispersion equation in the problem B for plane wave (2.1) have the following form

$$D_B(\boldsymbol{\alpha}, \omega) = \det\left[\mathbf{\Gamma}(\boldsymbol{\alpha}) - \rho\Omega^2(\boldsymbol{\alpha})\mathbf{E}\right] = 0, \tag{2.9}$$

where $\mathbf{E}$ is the unit matrix 2x2,

$$\Omega(\boldsymbol{\alpha}) = \omega + \mathbf{w} \cdot \boldsymbol{\alpha}. \tag{2.10}$$

The solution of equation (2.10) can be leaved in the form of the set of hypersurfaces

$$\omega = \omega_j^{B(k)}(\boldsymbol{\alpha}) = \alpha c_{pj}^{B(k)}(\mathbf{n}), \quad j = 1, 2. \tag{2.11}$$

From the dispersion surface (2.11) the group velocity vector $\mathbf{c}_{gj}^{B(k)} = \mathbf{c}_{gj}^{B(k)}(\mathbf{n})$ may be found by the following way

$$\mathbf{c}_{gj}^{B(k)} = \frac{\partial \omega_j^{B(k)}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} = \mathbf{c}_{pj}^{B(k)} + (\mathbf{E} - \mathbf{n}\mathbf{n}^*) \cdot \frac{\partial c_{pj}^{B(k)}}{\partial \mathbf{n}}, \qquad (2.12)$$

or, by using (2.7),

$$\mathbf{c}_{gj}^{B(k)} = (-1)^k \mathbf{c}_{gj}^A(\mathbf{n}) - \mathbf{w}. \qquad (2.13)$$

It is known [10], that for plane waves in the problem A these important relations are correct $(\mathbf{L}^A = \mathbf{n}/c_p^A)$

$$\mathbf{c}_g^A \cdot \mathbf{n} = c_p^A, \quad \mathbf{c}_g^A \cdot \mathbf{L}^A = 1, \quad \mathbf{n} \cdot d\mathbf{c}_g^A = 0, \quad \mathbf{c}_g^A \cdot d\mathbf{L}^A = 0.$$

As it is shown in [7] for the case of tree-dimensional problem, the analogical formulae are valid. This proof is completely transported on the considered plane problem. Therefore the following relations are correct

$$\mathbf{c}_{gj}^{B(k)} \cdot \mathbf{n} = c_{pj}^{B(k)};\ \mathbf{c}_{gj}^{B(k)} \cdot \mathbf{L}_j^{B(k)} = 1;\ \mathbf{n} \cdot d\mathbf{c}_{gj}^{B(k)} = 0;\ \mathbf{c}_{gj}^{B(k)} \cdot d\mathbf{L}_j^{B(k)} = 0. \qquad (2.14)$$

(In (2.14) and in the next the summation by repeating index is absent.)

After analyzing obtained formulae (2.14), we may establish for the problem B the basic properties of phase velocity curves $\mathbf{c}_{pj}^{B(k)}(\mathbf{n}) = c_{pj}^{B(k)}(\mathbf{n})\mathbf{n}$, slowness curves (inverse velocity or reflection curves) $\mathbf{L}_j^{B(k)}(\mathbf{n}) = \mathbf{n}/c_{pj}^{B(k)}(\mathbf{n})$ and group velocity curves (waves curves or ray curves) $\mathbf{c}_{gj}^{B(k)}(\mathbf{n})$. In general, all of these curves in the problem B with $w \neq 0$ do not have the central symmetry $\mathbf{n} \leftrightarrow (-\mathbf{n})$, and their crystallographical symmetry does not reserve. The phase velocity curves $\mathbf{c}_{pj}^{B(k)}$ and waves curves $\mathbf{c}_{gj}^{B(k)}$ are limited for all values w. But the slowness curves $\mathbf{L}_j^{B(k)}$ can be both limited and unlimited depending on motion behavior.

We shall call the source motion rate superseismic, if in R^2 the directions $\mathbf{n}$, along which there are no slowness curves, exist, and all slowness curves are unlimited. If there is one limited slowness curve, but also unlimited slowness curves exist, then we shall call this motion behavior transseismic.

We note that the slowness curves $\mathbf{L}_j^{B(k)}$ may essentially differ from corresponding curves for the problem A, and their number may vary from 2 to 4. At the same time, the group velocity curves $\mathbf{c}_{gj}^{B(k)}$ for the problem B may be obtained by simple transfer of curves for the problem A by vector $(-\mathbf{w})$. Besides, with trans- and superseismic motion rate two parts $\mathbf{c}_{gj}^{B0}$ and $\mathbf{c}_{gj}^{B1}$ form one closed curve. For this behavior in case (2.5) for fixed $\mathbf{n}$ and j we have two plane waves (fast and slow) with inverse velocities, which belong to two different slowness curves.

In addition to this from (2.2) and (2.14) important common properties follow [7]. Thus, the quasi-longitudinal and quasi-shear curves save their types for all $\mathbf{w}$, because the polarization vector $\mathbf{p}_j$ from (2.2) does not depend on the velocity $\mathbf{w}$.

The group velocity vector $\mathbf{c}_g$, by (2.14) in every point of slowness curve $\mathbf{L}$, is orthogonal to the tangent to $\mathbf{L}$ $(\mathbf{c}_g \cdot d\mathbf{L} = 0)$. Conversely, the wave normal vector $\mathbf{n}$ of plane wave with group velocity $\mathbf{c}_g(\mathbf{n})$ is orthogonal to tangent in the corresponding point of wave velocity $\mathbf{c}_g$ $(\mathbf{n} \cdot d\mathbf{c}_g = 0)$.

Everything considered above is illustrated by Figs. 1–11. These figures show slowness and group velocities curves for different kinds of quartz for plane deformations in planes Ox_1x_2 and Ox_2x_3. Curves marked with "1" and "2" correspond to waves with phase velocities $c_{p1}^{B(k)}$ and $c_{p2}^{B(k)}$ respectively. The waves are numbered so that for the problem A $c_{p1}^{A} < c_{p2}^{A}$. Therefore in the most cases waves with subscript "1" will be quasi-shear and waves with subscript "2" will be quasi-longitudinal.

The figures 1–6 show characteristic curves of plane waves for fused silica (SiO_2), which is isotropic material. The elasticity modules and density of fused silica as well as other materials were taken from [9].

The figures 1 and 2 show curves of slowness and group velocities of fused silica without motion ($w = 0$) respectively. As it can be seen from Fig. 1 and 2 when $w = 0$ these curves represent couples of concentric circumferences.

When the velocity is not equal to zero pictures of characteristic curves essentially change. For subseismic motion when $w_1 = 0.6 \cdot 10^3$ m/s; $w_2 = 0$ the curves slowness and group velocities are shown on Fig. 3, 4 respectively. Comparing Fig. 1 and Fig. 3 we can notice that the curves of slowness change their shape and structure when w increases. Meantime, as it was mentioned above, group velocity curves (Fig. 2 and Fig. 4) simply transfer on vector $-\mathbf{w}$ relatively to the center of coordinate system.

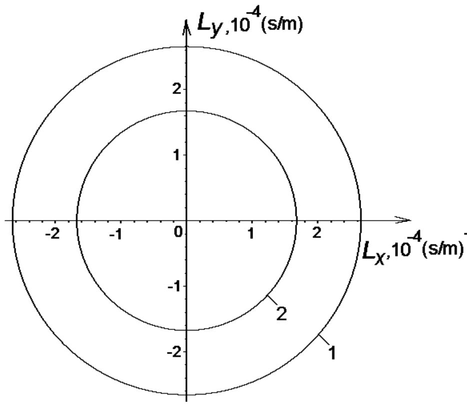


Fig. 1. Slowness curves, fused silica, $w_1 = w_2 = 0$.

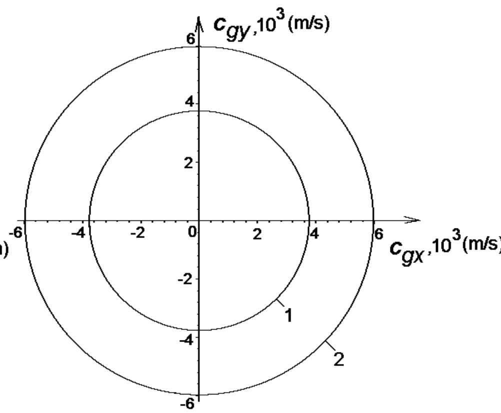


Fig. 2. Group velocity curves, fused silica, $w_1 = w_2 = 0$.

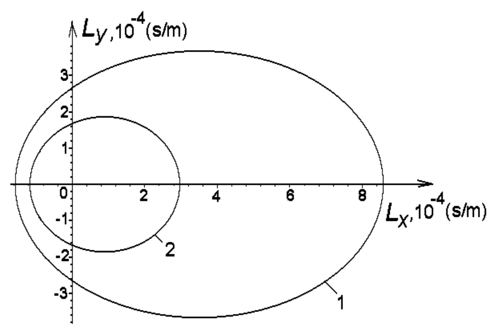


Fig. 3. Slowness curves, fused silica, $w_1 = 2600$, $w_2 = 0$.

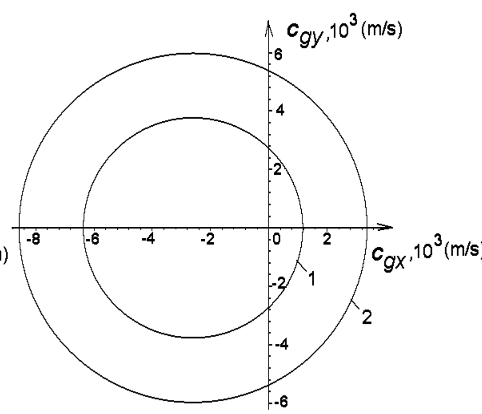
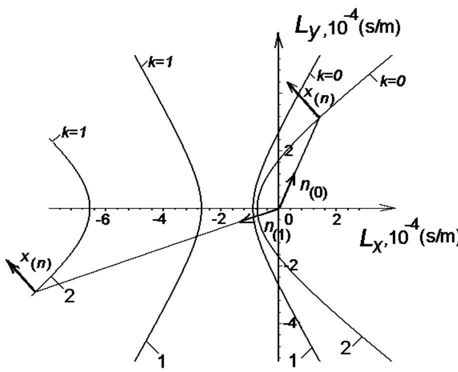


Fig. 4. Group velocity curves, fused silica, $w_1 = 2600$, $w_2 = 0$.



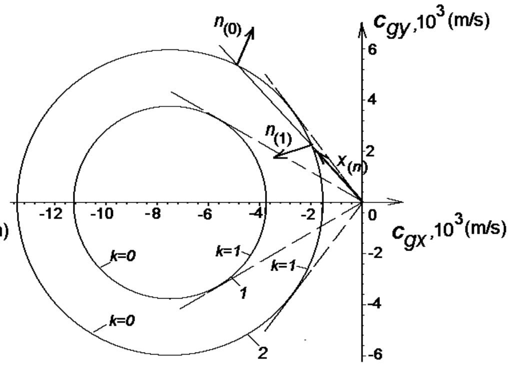


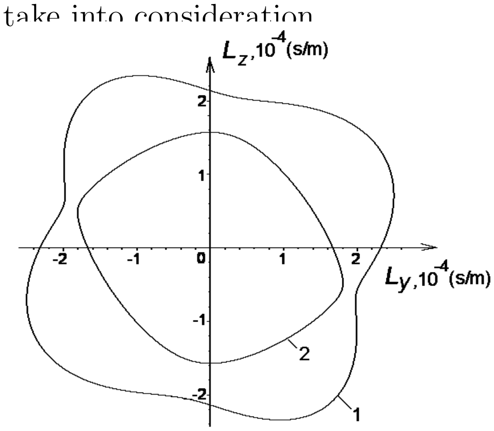
Fig. 5. Slowness curves, fused silica, $w_1 = 7500$, $w_2 = 0$.

Fig. 6. Group velocity curves, fused silica, $w_1 = 7500$, $w_2 = 0$.

The changes of the shape and structure of slowness curves and ultimate tendencies of these changes can be seen from Fig. 5, 6, where characteristic curves for superseismic motion when $w_1 = 7.5 \cdot 10^3$ m/s; $w_2 = 0$ are shown. Here Fig. 5 shows slowness curves and Fig. 6 shows group velocity curves. Slowness curves for superseismic motion (Fig. 5) are not limited. Each couple of these curves corresponds to fast and slow waves. Group velocity curves (Fig. 6) when $w_1 = 7. \cdot 10^3$ m/s; $w_2 = 0$ wholly move to the half-plane $x \leq 0$, this is caused by the absence of waves in front of the source for superseismic motion. The parts of fast and slow waves for group velocity curves are the pieces of the same curves separated by tangents to these curves which pass through the center of coordinate system (Fig. 6).

The figures 5 and 6 illustrate relations of orthogonality $\mathbf{c}_g \cdot d\mathbf{L} = 0$ and $\mathbf{L} \cdot d\mathbf{c}_g = \mathbf{n} \cdot d\mathbf{c}_g = 0$. Here slowness curves have directions $\mathbf{n}_{(0)}$ and $\mathbf{n}_{(0)}$ associated with one direction $\mathbf{x}_{(n)}$ on picture of group velocities, and vice versa.

For anisotropic materials the pictures of characteristic curves can be much more complicate. The Figures 7–10 show curves for slowness and group velocities curves for α-quartz (SiO_2) for plane deformation in the plane x_2x_3. This material relates to trigonal crystal system of 32 class and possesses piezoelectric properties which we don't take into consideration.



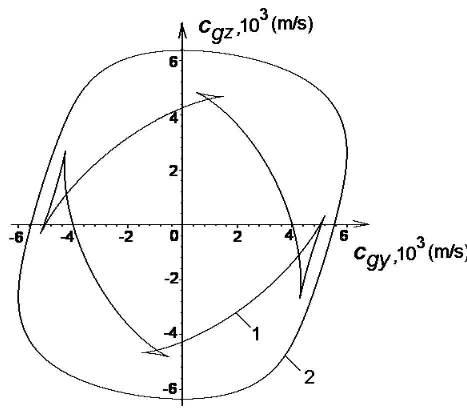


Fig. 7. Slowness curves, α-quartz, $w_1 = w_2 = 0$.

Fig. 8. Group velocity curves, α-quartz, $w_1 = w_2 = 0$.

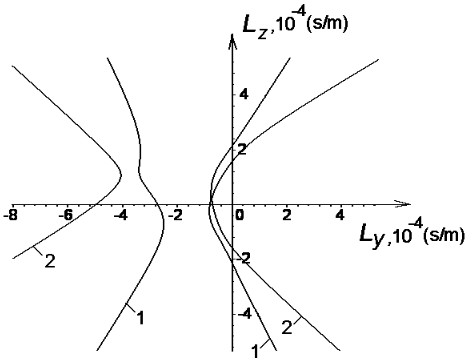


Fig. 9. Slowness curves, α-quartz, $w_1 = 8000$, $w_2 = 0$.

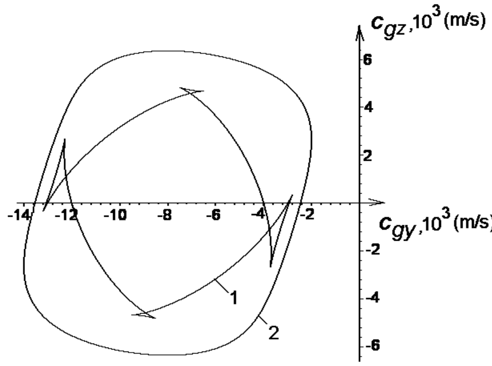


Fig. 10. Group velocity curves, α-quartz, $w_1 = 8000$, $w_2 = 0$.

Slowness curves for α-quartz on Fig. 7 and 9 have the points of inflection and the sections of convexity and concavity. Therefore group velocity curves have typical acute edges. Availability of such edges defines areas with different numbers of spreading waves even for problem A [11]. This property is also secured for problem B with additional sophistication of the wave picture and distraction of the curve central symmetry.

3 Fundamental solutions

The fundamental solution for the problem B is the solution of equation (1.9) with the point external source $\mathbf{f}$

$$\mathbf{f} = f\delta(\mathbf{x})\mathbf{l}\,\exp{(i\omega t)}.$$

For obtaining unique solution we use the principle of limiting absorption, according to which in the case of oscillation of point source by the law $\exp{(i\omega t)}$ the amplitude $\mathbf{v}(\mathbf{x})$ in equation (1.1) should be defined as a limit at $\varepsilon \to +0$

$$\mathbf{v} = \lim_{\varepsilon \to +0} \mathbf{v}_\varepsilon. \tag{3.1}$$

The function $\mathbf{v}_\varepsilon$ satisfies the following ε-problem

$$-\mathbf{L}^*(\nabla^x) \cdot \mathbf{C} \cdot \mathbf{L}(\nabla^x) \cdot \mathbf{v}_\varepsilon + \rho(i\omega_\varepsilon - \mathbf{w} \cdot \nabla^x)^2 \mathbf{v}_\varepsilon = f\delta(\mathbf{x})\mathbf{l}, \tag{3.2}$$

where $\omega_\varepsilon = \omega - i\varepsilon$, $\varepsilon > 0$, $\varepsilon \ll 1$.

By applying Fourier integral transforms along x_1, x_2, we obtain the fundamental solution $\mathbf{v}_\varepsilon$ for equation (3.2) in integral form

$$\mathbf{v}_\varepsilon(\mathbf{x}) = \frac{f}{4\pi^2} \iint_{R^2} \mathbf{K}^{-1}(\boldsymbol{\alpha}) \cdot \mathbf{l}\,\mathrm{e}^{-i\boldsymbol{\alpha}\cdot\mathbf{x}}\, d\alpha_1\, d\alpha_2, \tag{3.3}$$

where

$$\mathbf{K}(\boldsymbol{\alpha}) = \boldsymbol{\Gamma}(\boldsymbol{\alpha}) - \rho\Omega_\varepsilon^2(\boldsymbol{\alpha})\mathbf{E}, \tag{3.4}$$

$$\Omega_\varepsilon(\boldsymbol{\alpha}) = \omega_\varepsilon + \mathbf{w} \cdot \boldsymbol{\alpha}. \tag{3.5}$$

Introducing the polar coordinate system $(\alpha,\, \theta)$

$$\boldsymbol{\alpha} = \alpha \mathbf{n}, \quad \mathbf{n} = \{\cos\theta;\, \sin\theta\}, \tag{3.6}$$

it is possible to decompose the matrix $\mathbf{K}^{-1}(\boldsymbol{\alpha})$ by accompanying matrices $\mathbf{P}_j$ [12], which are composed of eigenvectors $\mathbf{p}_1$, $\mathbf{p}_2$ for problem (2.2)

$$\mathbf{K}^{-1}(\boldsymbol{\alpha}) = \frac{1}{\rho}\sum_{j=1}^{2}\frac{\mathbf{P}_j}{\alpha^2\nu_j^2 - \Omega_\varepsilon^2(\boldsymbol{\alpha})}, \tag{3.7}$$

$$\mathbf{P}_j = \mathbf{P}_j(\mathbf{n}) = \mathbf{p}_j(\mathbf{n})\mathbf{p}_j^*(\mathbf{n}), \quad \mathbf{n} = \mathbf{n}(\theta), \tag{3.8}$$

with regard to (3.6)—(3.8) we obtain from (3.3)

$$\mathbf{v}_\varepsilon = \frac{f}{4\pi^2\rho}\sum_{j=1}^{2}\int_0^{2\pi}\mathbf{P}_j\cdot\mathbf{l}\, I_{j\varepsilon}\, d\theta, \tag{3.9}$$

$$I_{j\varepsilon} = \int_0^{+\infty}\frac{\alpha}{\alpha^2\nu_j^2(\mathbf{n}) - \Omega_\varepsilon^2(\alpha\mathbf{n})}\,\mathrm{e}^{-iz\alpha}\, d\alpha, \quad z = \mathbf{n}\cdot\mathbf{x}. \tag{3.10}$$

We decompose integrand function from (3.10) by partial fractions

$$\frac{\alpha}{\alpha^2\nu_j^2(\mathbf{n}) - \Omega_\varepsilon^2(\alpha\mathbf{n})} = \frac{1}{2\omega_\varepsilon\nu_j}\left(\frac{\alpha_{j\varepsilon}^+}{\alpha - \alpha_{j\varepsilon}^+} - \frac{\alpha_{j\varepsilon}^-}{\alpha - \alpha_{j\varepsilon}^-}\right), \tag{3.11}$$

$$\alpha_{j\varepsilon}^+ = \frac{\omega_\varepsilon}{\nu_j - w_n}, \quad \alpha_{j\varepsilon}^- = -\frac{\omega_\varepsilon}{\nu_j + w_n}. \tag{3.12}$$

In the integral (3.9) from θ we realize the following transformations

$$\int\limits_0^{2\pi}(\ldots)(\theta)\, d\theta = \int\limits_{-\pi/2+\tilde\theta}^{\pi/2+\tilde\theta}(\ldots)(\theta)\, d\theta + \int\limits_{-\pi/2+\tilde\theta}^{\pi/2+\tilde\theta}(\ldots)(\theta+\pi)\, d\theta, \tag{3.13}$$

where $\tilde\theta$ is the angle in the polar coordinate system for physical plane $\mathbf{x}$

$$\mathbf{x} = r\mathbf{y}, \quad |\mathbf{y}| = 1, \quad \mathbf{y} = \{\cos\tilde\theta, \sin\tilde\theta\}. \tag{3.14}$$

By taking into account (3.11)—(3.13) the formulae (3.9), (3.10) can be transformed in the following way

$$\mathbf{v}_\varepsilon(\mathbf{x}) = \frac{f}{8\pi^2\rho\omega_\varepsilon}\sum_{j=1}^{2}\int\limits_{-\pi/2+\tilde\theta}^{\pi/2+\tilde\theta}\frac{\mathbf{P}_j(\mathbf{n})\cdot\mathbf{l}}{\nu_j(\mathbf{n})}\, J_{j\varepsilon}(\theta)\, d\theta, \tag{3.15}$$

$$J_{j\varepsilon}(\theta) = \alpha_{j\varepsilon}^+\left(\int_0^{+\infty}\frac{\mathrm{e}^{-iz\alpha}}{\alpha - \alpha_{j\varepsilon}^+}\, d\alpha + \int_0^{+\infty}\frac{\mathrm{e}^{iz\alpha}}{\alpha + \alpha_{j\varepsilon}^+}\, d\alpha\right) - \tag{3.16}$$

$$-\alpha_{j\varepsilon}^{-}\left(\int_{0}^{+\infty}\frac{\mathrm{e}^{-iz\alpha}}{\alpha-\alpha_{j\varepsilon}^{-}}\,d\alpha+\int_{0}^{+\infty}\frac{\mathrm{e}^{iz\alpha}}{\alpha+\alpha_{j\varepsilon}^{-}}\,d\alpha\right).$$

The calculation of integrals from (3.16) depends on real and imaginary parts of $\alpha_{j\varepsilon}^{\pm}$, which in their turn depend on moving rate

$$\begin{aligned}
&\operatorname{Re}\alpha_{j\varepsilon}^{+}>0,\ \operatorname{Im}\alpha_{j\varepsilon}^{+}\le 0, \quad \nu_j-w_n>0,\\
&\operatorname{Re}\alpha_{j\varepsilon}^{+}<0,\ \operatorname{Im}\alpha_{j\varepsilon}^{+}\ge 0, \quad \nu_j-w_n<0,\\
&\operatorname{Re}\alpha_{j\varepsilon}^{-}<0,\ \operatorname{Im}\alpha_{j\varepsilon}^{-}\ge 0, \quad \nu_j+w_n>0,\\
&\operatorname{Re}\alpha_{j\varepsilon}^{-}>0,\ \operatorname{Im}\alpha_{j\varepsilon}^{-}\le 0, \quad \nu_j+w_n<0.
\end{aligned} \tag{3.17}$$

Besides, it is essentially, that $\forall\theta\in[-\pi/2+\tilde\theta,\pi/2+\tilde\theta]$

$$z=\mathbf{n}\cdot\mathbf{x}=r\cos(\theta-\tilde\theta)>0. \tag{3.18}$$

We shall introduce the contours $C_\Gamma^{\pm}=[0,R]\cup C_R^{\pm}\cup[\pm iR,0]$, where C_R^{+} and C_R^{-} are the quoters of circle with radius R and center in the origin of coordinate system, lying respectively in the first and in the fourth quadrant of complex plane α, $R\gg 1$.

Calculating the integrals by complex integration method, using contours $C_\Gamma^{\pm}$, relations (3.17), (3.18) and Gordan's lemma and turned R to infinity, we obtain ($z>0$)

$$\int_{0}^{+\infty}\frac{\mathrm{e}^{-iz\alpha}}{\alpha-\alpha_{j\varepsilon}^{\pm}}\,d\alpha=-2\pi i\mathrm{e}^{-iz\alpha_{j\varepsilon}^{\pm}}H[\pm\nu_j-w_n]+i\int_{0}^{+\infty}\frac{(\alpha_{j\varepsilon}^{\pm}-i\eta)}{(\alpha_{j\varepsilon}^{\pm})^2+\eta^2}\mathrm{e}^{-z\eta}\,d\eta, \tag{3.19}$$

$$\int_{0}^{+\infty}\frac{\mathrm{e}^{iz\alpha}}{\alpha+\alpha_{j\varepsilon}^{\pm}}\,d\alpha=i\int_{0}^{+\infty}\frac{(\alpha_{j\varepsilon}^{\pm}-i\eta)}{(\alpha_{j\varepsilon}^{\pm})^2+\eta^2}\mathrm{e}^{-z\eta}\,d\eta,$$

where H is the Heaviside function.

On substitution (3.19) into (3.16) and received formula into (3.15) and implementing passage to the limit when $\varepsilon\to+0$, we obtain the final representation of fundamental solutions for the problems A and B with arbitrary moving rate

$$\mathbf{v}=\mathbf{v}_d+\mathbf{v}_0, \tag{3.20}$$

$$\mathbf{v}_d=\frac{if}{4\pi\rho}\sum_{j=1}^{2}\Big({\sum_k}'\Big)(-1)^{k+1}I_{jk}, \tag{3.21}$$

$$I_{jk}=\int\limits_{-\pi/2+\tilde\theta}^{\pi/2+\tilde\theta}\frac{\mathbf{P}_j(\theta)\cdot\mathbf{l}\,H[(-1)^k\nu_j-w_n]}{\nu_j(\theta)c_{pj}^{B(k)}(\theta)}\,\mathrm{e}^{-i\omega\mathbf{L}_j^{B(k)}\cdot\mathbf{x}}\,d\theta, \tag{3.22}$$

$$\mathbf{v}_0=\frac{if}{4\pi^2\rho\omega}\sum_{j=1}^{2}\int\limits_{-\pi/2+\tilde\theta}^{\pi/2+\tilde\theta}\frac{\mathbf{P}_j(\theta)\cdot\mathbf{l}}{\nu_j(\theta)}\,I_j^0\,d\theta, \tag{3.23}$$

$$I_j^0 = \int\limits_0^{+\infty} \left(\alpha_j^+ \frac{\alpha_j^+ - i\eta}{(\alpha_j^+)^2 + \eta^2} - \alpha_j^- \frac{\alpha_j^- - i\eta}{(\alpha_j^-)^2 + \eta^2} \right) \mathrm{e}^{-z\eta}\, d\eta, \tag{3.24}$$

$$z = \mathbf{n} \cdot \mathbf{x} = r\cos(\theta - \tilde{\theta}), \quad \alpha_j^\pm = \omega/(\pm\nu_j - w_n), \tag{3.25}$$

where symbol $\sum_k'$ in (3.21) denotes the presence or the absence of summation by k according to moving rate.

In addition formula (3.12) can be transformed by using the following integrals

$$\int_0^{+\infty} \frac{\mathrm{e}^{-z\eta}}{\zeta^2 + \eta^2}\, \eta\, d\eta = -[\mathrm{ci}\,(\zeta z)\cos(\zeta z) + \mathrm{si}\,(\zeta z)\sin(\zeta z)], \tag{3.26}$$

$$\int_0^{+\infty} \frac{\mathrm{e}^{-z\eta}}{\zeta^2 + \eta^2}\, d\eta = \frac{1}{\zeta}[\mathrm{ci}\,(\zeta z)\sin(\zeta z) - \mathrm{si}\,(\zeta z)\cos(\zeta z)],$$

where $\mathrm{si}\,(x)$ and $\mathrm{ci}\,(x)$ are respectively integral sine and cosine.

For the problem A with $w = 0$ the fundamental solutions representations in form (3.21)—(3.26) can be significantly simplified

$$\mathbf{v}_d = \frac{if}{4\pi\rho} \sum_{j=1}^{2} \int\limits_{-\pi/2+\tilde{\theta}}^{\pi/2+\tilde{\theta}} \frac{\mathbf{P}_j(\theta) \cdot \mathbf{l}}{\nu_j^2(\theta)}\, \mathrm{e}^{-i\omega \mathbf{L}_j^A \cdot \mathbf{x}}\, d\theta, \tag{3.27}$$

$$\mathbf{v}_0 = -\frac{if}{2\pi^2\rho} \sum_{j=1}^{2} \int\limits_{-\pi/2+\tilde{\theta}}^{\pi/2+\tilde{\theta}} \frac{\mathbf{P}_j(\theta) \cdot \mathbf{l}}{\nu_j^2(\theta)} [\mathrm{ci}\,(\frac{\omega}{\nu_j}z)\cos(\frac{\omega}{\nu_j}z) + \mathrm{si}\,(\frac{\omega}{\nu_j}z)\sin(\frac{\omega}{\nu_j}z)]\, d\theta.$$

Thus, we have obtained the fundamental solutions for problem A and B in the form of integrals by finite interval. It should be noted that the representation (3.20)—(3.26) for the problem B are suitable for arbitrary moving rate.

4 Far field asymptotics

The term $\mathbf{v}_d$ from (3.21), (3.22) or (3.27) define dynamic effect of displacement field. We shall determine the asymptotics of $\mathbf{v}_d$ in far field $\omega r >> 1$ for general case of the problem B. For this it is evidently required to find the asymptotics of oscillating integral (3.22).

From classical method of stationary phase the contribution of individual stationary point θ_s for integral

$$\int F(\theta)\mathrm{e}^{iq(\theta)\omega r}\, d\theta \tag{4.1}$$

at $\omega r >> 1$ can be given by the expression

$$\sqrt{\frac{2\pi}{\omega r |q''(\theta_s)|}} F(\theta_s) \mathrm{e}^{i[q(\theta_s)\omega r + \frac{\pi}{4}\mathrm{sign}\, q''(\theta_s)]}\, d\theta, \tag{4.2}$$

where θ_s is the saddle (stationary) point, or the root of equation

$$q'(\theta_s) = 0. \tag{4.3}$$

In our problem for fixed j and k we have

$$F(\theta) = \frac{\mathbf{P}_j(\theta)\cdot \mathbf{l}\, H[(-1)^k \nu_j - w_n]}{\nu_j(\theta) c_{pj}^{B(k)}(\theta)}, \quad q(\theta) = -\mathbf{L}_j^{B(k)} \cdot \mathbf{y}. \tag{4.4}$$

We shall obtain the formula for saddle point θ_s. From (4.3), (4.4) we have

$$\frac{\partial q}{\partial \theta} = -\frac{\partial \mathbf{L}_j^{B(k)}}{\partial \theta} \cdot \mathbf{y}. \tag{4.5}$$

We can represent the components of vector $\partial \mathbf{L}_j^{B(k)}/\partial\theta$ in the form

$$\frac{\partial L_{ji}^{B(k)}}{\partial \theta} = \sum_{m=1}^{2} \frac{\partial L_{ji}^{B(k)}}{\partial n_m} \frac{\partial n_m}{\partial \theta},$$

and since

$$\frac{\partial L_{ji}^{B(k)}}{\partial n_m} = \frac{\partial}{\partial n_m}\left(\frac{n_i}{c_{pj}^{B(k)}}\right) = \frac{1}{c_{pj}^{B(k)}}\left(\delta_{im} - L_{ji}^{B(k)} \frac{\partial c_{pj}^{B(k)}}{\partial n_m}\right),$$

it follows

$$\frac{\partial \mathbf{L}_j^{B(k)}}{\partial \theta} = \frac{1}{c_{pj}^{B(k)}}\left[\mathbf{E} - \mathbf{L}_j^{B(k)}\left(\frac{\partial c_{pj}^{B(k)}}{\partial \mathbf{n}}\right)^*\right] \cdot \frac{\partial \mathbf{n}}{\partial \theta}. \tag{4.6}$$

Then by using (2.12) we find

$$\mathbf{c}_{gj}^{B(k)} \cdot \frac{\partial \mathbf{n}}{\partial \theta} = \frac{\partial c_{pj}^{B(k)}}{\partial \mathbf{n}} \cdot \frac{\partial \mathbf{n}}{\partial \theta}, \tag{4.7}$$

since $\mathbf{c}_{pj}^{B(k)} \cdot (\partial \mathbf{n}/\partial\theta) = 0$, $\mathbf{n} \cdot (\partial \mathbf{n}/\partial \theta) = 0$.

From (4.7) the formula (4.6) can be rewritten in the form

$$\frac{\partial \mathbf{L}_j^{B(k)}}{\partial \theta} = \frac{1}{c_{pj}^{B(k)}}\left[\mathbf{E} - \mathbf{L}_j^{B(k)}(\mathbf{c}_{gj}^{B(k)})^*\right] \cdot \frac{\partial \mathbf{n}}{\partial \theta}. \tag{4.8}$$

On substitution of (4.8) into (4.5), (4.3), we obtain

$$\mathbf{y}^* \cdot \left[\mathbf{E} - \mathbf{L}_j^{B(k)}(\mathbf{c}_{gj}^{B(k)})^*\right] \cdot \frac{\partial \mathbf{n}}{\partial \theta} = 0. \tag{4.9}$$

Besides, using one of the relations (2.14) ($\mathbf{L}_j^{B(k)} \cdot \mathbf{c}_{gj}^{B(k)} = 1$) we can write following sequence of equalities

$$0 = \mathbf{n}\cdot\mathbf{y} - (\mathbf{L}_j^{B(k)} \cdot \mathbf{c}_{gj}^{B(k)})\mathbf{n}\cdot\mathbf{y} = \mathbf{n}^* \cdot \mathbf{E} \cdot \mathbf{y} - (\mathbf{n}\cdot \mathbf{c}_{gj}^{B(k)})\frac{\mathbf{n}}{c_{pj}^{B(k)}} \cdot \mathbf{y} =$$

$$= \mathbf{n}^* \cdot \mathbf{E} \cdot \mathbf{y} - \mathbf{n}^* \cdot (\mathbf{c}_{gj}^{B(k)}(\mathbf{L}_j^{B(k)})^*) \cdot \mathbf{y} = \mathbf{y}^* \cdot [\mathbf{E} - \mathbf{L}_j^{B(k)}(\mathbf{c}_{gj}^{B(k)})^*] \cdot \mathbf{n},$$

i.e.

$$\mathbf{y}^* \cdot [\mathbf{E} - \mathbf{L}_j^{B(k)}(\mathbf{c}_{gj}^{B(k)})^*] \cdot \mathbf{n} = 0. \tag{4.10}$$

From (4.9), (4.10) we obtain, that the vector $\mathbf{y}^* \cdot [\mathbf{E} - \mathbf{L}_j^{B(k)}(\mathbf{c}_{gj}^{B(k)})^*]$ is orthogonal for two one-to-one orthogonal vectors $\mathbf{n}$ and $\partial\mathbf{n}/\partial\theta$, and therefore, it is equal to zero

$$\mathbf{y}^* \cdot [\mathbf{E} - \mathbf{L}_j^{B(k)}(\mathbf{c}_{gj}^{B(k)})^*] = 0. \tag{4.11}$$

By using inequality (3.18), which can be rewritten in the form

$$\mathbf{y} \cdot \mathbf{L}_j^{B(k)}(\theta) > 0, \tag{4.12}$$

we can get the following shape for the equation (4.11)

$$\mathbf{c}_{gj}^{B(k)}(\theta)\,|\mathbf{y} \cdot \mathbf{L}_j^{B(k)}(\theta)| = \mathbf{y}. \tag{4.13}$$

The relations (4.13), (4.12) are the suitable formulae for definitions of stationary points $\theta_{jm}^{(k)}$ and wave normal vector $\mathbf{n}_{jm}^{(k)} = \mathbf{n}(\theta_{jm}^{(k)})$.

Geometrically, from (4.12), (4.13), just as in problem A, for stationary points the group velocity vector $\mathbf{c}_{gjm}^{B(k)} = \mathbf{c}_{gj}^{B(k)}(\mathbf{n}_{jm}^{(k)})$, which is perpendicular to slowness curve $\mathbf{L}_j^{B(k)}$, is directed along vector $\mathbf{y}$ ($\mathbf{x}$). Therefore, for fixed direction $\mathbf{x}$ on physical plane the stationary value $\mathbf{n}_{jm}^{(k)}$ will appear such values $\mathbf{n}$, in which the external normal to slowness curve coincides with direction $\mathbf{x}$.

Now, we shall dwell upon calculation the expression q'' in the stationary point $\theta_{jm}^{(k)}$

$$q'' = -\mathbf{y} \cdot \frac{\partial^2 \mathbf{L}_j^{B(k)}}{\partial\theta^2}. \tag{4.14}$$

Let $\boldsymbol{\tau}$ be tangent to slowness curve $\mathbf{L}_j^{B(k)}$ unit vector and s be natural parameter along this curve. We note, that $\mathbf{y}$ is the unit normal vector to $\mathbf{L}_j^{B(k)}$, and thus, $\mathbf{y} \cdot \partial\mathbf{L}_j^{B(k)}/\partial\theta = 0$.

Because

$$\frac{\partial\mathbf{y}}{\partial\theta} \cdot \frac{\partial\mathbf{L}_j^{B(k)}}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\mathbf{y} \cdot \frac{\partial\mathbf{L}_j^{B(k)}}{\partial\theta}\right) - \mathbf{y} \cdot \frac{\partial^2\mathbf{L}_j^{B(k)}}{\partial\theta^2},$$

then we can transform formulae (4.14) in the form

$$q'' = \frac{\partial\mathbf{y}}{\partial\theta} \cdot \frac{\partial\mathbf{L}_j^{B(k)}}{\partial\theta}. \tag{4.15}$$

By using Frene formulae, we can write

$$\frac{\partial\mathbf{L}_j^{B(k)}}{\partial\theta} = \frac{\partial\mathbf{L}_j^{B(k)}}{\partial s}\frac{\partial s}{\partial\theta} = \boldsymbol{\tau}\frac{\partial s}{\partial\theta}, \quad \frac{\partial\mathbf{y}}{\partial\theta} = \frac{\partial\mathbf{y}}{\partial s}\frac{\partial s}{\partial\theta} = -k_{pj}^{B(k)}\boldsymbol{\tau}\frac{\partial s}{\partial\theta}, \tag{4.16}$$

where $k_{pj}^{B(k)}$ is the curvature of slowness curve $\mathbf{L}_j^{B(k)}$.

From (4.15), (4.16) we have

$$q'' = -k_{pj}^{B(k)} \left(\frac{\partial s}{\partial \theta}\right)^2 = -k_{pj}^{B(k)} \frac{\partial \mathbf{L}_j^{B(k)}}{\partial \theta} \cdot \frac{\partial \mathbf{L}_j^{B(k)}}{\partial \theta}. \tag{4.17}$$

By using (4.8), it is easy to find, that

$$\frac{\partial \mathbf{L}_j^{B(k)}}{\partial \theta} \cdot \frac{\partial \mathbf{L}_j^{B(k)}}{\partial \theta} = \frac{1}{(c_{pj}^{B(k)})^4}[(c_{pj}^{B(k)})^2 + (\mathbf{c}_{gj}^{B(k)} \cdot \frac{\partial \mathbf{n}}{\partial \theta})^2]. \tag{4.18}$$

By decomposing group velocity vector $\mathbf{c}_{gj}^{B(k)}$ in system of orthonormalized vector $\mathbf{n}$, $\partial \mathbf{n}/\partial \theta$, we have

$$\mathbf{c}_{gj}^{B(k)} = c_{gjn}^{B(k)} \mathbf{n} + c_{gj\theta}^{B(k)} \frac{\partial \mathbf{n}}{\partial \theta},$$

with $c_{gjn}^{B(k)} = c_{pj}^{B(k)}$, $c_{gj\theta}^{B(k)} = \mathbf{c}_{gj}^{B(k)} \cdot (\partial \mathbf{n}/\partial \theta)$.
Therefore,

$$|\mathbf{c}_{gj}^{B(k)}|^2 = (c_{pj}^{B(k)})^2 + \left(\mathbf{c}_{gj}^{B(k)} \cdot \frac{\partial \mathbf{n}}{\partial \theta}\right)^2,$$

and as the result we obtain from (4.17), (4.18) the simple expression

$$q'' = -k_{pj}^{B(k)} \frac{(c_{gj}^{B(k)})^2}{(c_{pj}^{B(k)})^4}. \tag{4.19}$$

By substituting of (4.19) into (4.2), (4.1), (3.27) yields the following formulae for far field asymptotic

$$\mathbf{v}_d = \sum_{j=1}^{2} (\sum_k)' \sum_{m=1}^{N_j^{(k)}} \mathbf{v}_{jm}^{(k)}, \quad \omega r \to \infty, \tag{4.20}$$

$$\mathbf{v}_{jm}^{(k)} \approx (-1)^{k+1} \frac{if \mathbf{P}_j(\mathbf{n}_{jm}^{(k)}) \cdot \mathbf{l}\, c_{pjm}^{B(k)}}{2\sqrt{2\pi r}\rho |\mathbf{c}_{gjm}^{B(k)}| c_{pjm}^{A} \sqrt{|k_{pjm}^{B(k)}|}} \mathrm{e}^{-i(\omega r \mathbf{L}_{jm}^{B(k)} \cdot \mathbf{y} + \frac{\pi}{4}\sigma_{jm}^{B(k)})}, \tag{4.21}$$

where $N_j^{(k)}$ is the number of stationary point for separated slowness curve, $\mathbf{L}_{jm}^{B(k)} = \mathbf{L}_j^{B(k)}(\mathbf{n}_{jm}^{(k)})$, $\mathbf{c}_{gjm}^{B(k)} = \mathbf{c}_{gj}^{B(k)}(\mathbf{n}_{jm}^{(k)})$, $\mathbf{c}_{pjm}^{B(k)} = \mathbf{c}_{pj}^{B(k)}(\mathbf{n}_{jm}^{(k)})$, $\mathbf{c}_{pjm}^{A} = \mathbf{c}_{pj}^{A}(\mathbf{n}_{jm}^{(k)})$, $k_{pjm}^{B(k)} = k_{pj}^{B(k)}(\mathbf{n}_{jm}^{(k)})$, $\sigma_{jm}^{B(k)} = \operatorname{sign} k_{pjm}^{B(k)}$.
Because of $(-1)^k c_{pjm}^{B(k)}/c_{pjm}^{A} = \omega/\Omega(\boldsymbol{\alpha}_{jm}^{(k)})$, and in the stationary points from (4.12), (4.13) $\mathbf{L}_{jm}^{B(k)} \cdot \mathbf{y} = |\mathbf{c}_{gjm}^{B(k)}|^{-1}$, then we can rewrite formula (4.21) in the form

$$\mathbf{v}_{jm}^{(k)} \approx (-1)^{k+1} \frac{if\omega \mathbf{P}_j(\mathbf{n}_{jm}^{(k)}) \cdot \mathbf{l}}{2\sqrt{2\pi r}\rho \Omega(\boldsymbol{\alpha}_{jm}^{(k)}) |\mathbf{c}_{gjm}^{B(k)}| \sqrt{|k_{pjm}^{B(k)}|}} \mathrm{e}^{-i\left(\frac{\omega r}{|\mathbf{c}_{gjm}^{B(k)}|} + \frac{\pi}{4}\sigma_{jm}^{B(k)}\right)}. \tag{4.22}$$

As we can see from (4.21), (4.22), in far zone the wave field is separated in individual cylindrical waves $j = 1, 2$; $k = 0, 1$ or $k = 0$; $m = 1, ..., N_j^{(k)}$.

We note that the wave fields in the far zone have peculiarities in the neighborhood of directions $\mathbf{n}^{(k)}_{jm}(\mathbf{x})$ for which $k^{(k)}_{pjm} \approx 0$ for near located and for multiple points. For such directions as in the problem A [13] another shapes of asymptotic decomposition than (4.20)–(4.22) are required.

As it follows from (4.12), (4,13), the number of waves $\mathbf{v}^{(k)}_{jm}$ in far field are determined by the number of stationary value $\mathbf{n}^{(k)}_{jm}(\mathbf{x})$. As we marked above, in the stationary value $\mathbf{n}^{(k)}_{jm}(\mathbf{x})$ the group velocity vector $\mathbf{c}^{B(k)}_{gjm}(\mathbf{n}^{(k)}_{jm})$ is directed along the vector $\mathbf{x}$. Therefore, for fixed direction $\mathbf{x}$ the number of waves is easy evaluated as the number of intersection between the ray $0\mathbf{x}$ and the group velocity curves. Depending on moving rate, the number of waves may be essentially changed, and besides in the case of trans- and superseismic motion the zones of a fast and slow waves propagation, limited by Mach's cones, exist.

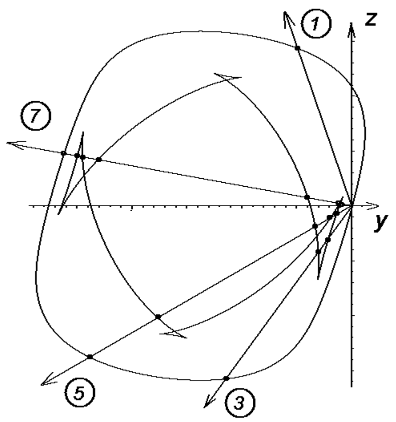


Fig. 11. Group velocity curves, α-quartz, $w_1 = 5500$, $w_2 = 0$.

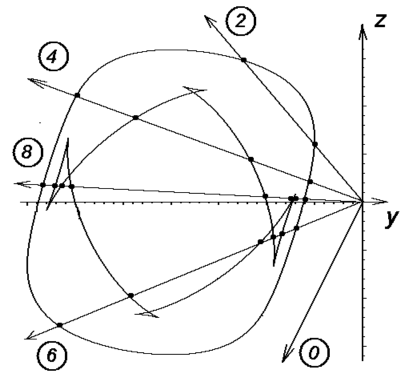


Fig. 12. Group velocity curves, α-quartz, $w_1 = 8000$, $w_2 = 0$.

The Figures 11, 12 show typical pictures of group velocity curves in planes which correspond to the work cartesian coordinate system. Separately the selected directions $\mathbf{x}$ on the Figs. 11, 12 are supplied with arrows and marked with numbers in circles which denote the number of waves for these directions, i.e. the number of intersection points of the curves c^A_{gj} and $c^{B(k)}_{gj}$ with directions $\mathbf{x}$. On the figures the points of intersection are bold-face points. Figs. 11, 12 illustrate, that the number of waves in different zones takes any values from 0 to 8.

In pole coordinate system $(r,\ \tilde{\theta})$ the components of group velocity vector in stationary points are given by: $c^B_{gjmr} = |\mathbf{c}^B_{gjm}|$; $c^B_{gjm\tilde{\theta}} = 0$. Hence, cylindrical waves (4.21), (4.22) in far field satisfy all general conditions for cylindrical waves from [3]. Therefore, for waves (4.21) or (4.22) the general energetic relations are correct [3], and the group velocity vector $\mathbf{c}^B_{gjm}$ is equal to energy transport velocity vector (ray velocity vector) for fixed and moving observers.

We use general formulae [3] for the cylindrical waves average energies $\langle E^{\xi x}\rangle$ and $\langle E^{x}\rangle$ in moving coordinate system for fixed and moving observers respectively

$$\langle E^{\xi x}\rangle = \frac{1}{2}\Omega^2(\boldsymbol{\alpha})\rho\mathbf{v}\cdot\mathbf{v}^*, \tag{4.23}$$

$$\langle E^{x}\rangle = \frac{1}{2}\omega\Omega(\boldsymbol{\alpha})\rho\mathbf{v}\cdot\mathbf{v}^*, \tag{4.24}$$

and formulae for average energy flux vector (Poynting's vector)

$$\langle J_r^{\xi x(x)} \rangle = c_{gr}^B \langle E^{\xi x(x)} \rangle, \tag{4.25}$$

where $\langle ... \rangle = \frac{1}{T} \int_0^T (...) \, dt$, $T = 2\pi/\omega$.

By substituting of (4.22) into (4.23)—(4.25) yields the following expressions for energy flux of individual cylindrical waves in far field

$$\langle J_{jmr}^{\xi x(k)} \rangle = \frac{\omega^2 f^2 |\mathbf{P}_j(\mathbf{n}_{jm}^{(k)}) \cdot \mathbf{l}|^2}{16\pi\rho r |\mathbf{c}_{gjm}^{B(k)}||k_{pjm}^{B(k)}|}, \tag{4.26}$$

$$\langle J_{jmr}^{x(k)} \rangle = \frac{\omega}{\Omega(\boldsymbol{\alpha}_{jm}^{(k)})} \langle J_{jmr}^{\xi x(k)} \rangle = (-1)^k \frac{c_{pjm}^{B(k)}}{c_{pjm}^{A}} \langle J_{jmr}^{\xi x(k)} \rangle. \tag{4.27}$$

For superseismic moving rate $\Omega(\boldsymbol{\alpha}_{jm}^1) < 0$ for $k = 1$, and therefore $\langle J_{jmr}^{x1} \rangle < 0$, i.e. the slow waves transfer the negative energy, measured by moving observer. But this property is common for slow waves in problem B with superseismic moving sources.

The Figures 13–22 show the average energy flux curves $\langle J_r^{\xi x(k)} \rangle$ for fused silica and α-quartz (SiO_2). The normalizing factor ζ for the values of average energy flux is equal to $\omega^2 f^2/(16\pi\rho r)$.

The energy flux curves have more complicated forms, then slowness and group velocity curves. Really, according to (4.26) the values of energy flux depend on the values of group velocity, curvature of slowness curve and form the scalar product between polarization vector $\mathbf{p}_{jm}^{(k)} = \mathbf{p}_j(\mathbf{n}_{jm}^{(k)})$ and the unit vector of source direction $\mathbf{l}$ (by (3.8) $|\mathbf{P}_j(\mathbf{n}_{jm}^{(k)}) \cdot \mathbf{l}| = |\mathbf{p}_j(\mathbf{n}_{jm}^{(k)})|$). For some angle θ the polarization vectors can be orthogonal to unit vector of source direction $\mathbf{l}$, because quasi-longitudinal and quasi-shear polarization vectors are perpendicular, and if the angle θ change by 2π, the polarization vectors also rotate by 2π. In these cases the energy flux vector is equal to zero. On the other hand the energy flux tends to infinity, when then curvature of slowness curve tends to zero. This exists, when slowness curves stretch on infinity and in the points of inflection. As a result, taking into account the possibility of existence of zones with different number of spread waves, the energy flux curves are more complicated and complex for analysis.

The Figures 13–18 show the results of energy flux calculations for isotropic material fused silica. The values of source motion are equal to corresponding values for Figs. 1-6 (for Figs. 13, 14 $w_1 = w_2 = 0$, for Figs. 15, 16 $w_1 = 2.6 \cdot 10^3$ m/s, for Figs. 17, 18 $w_1 = 7.5 \cdot 10^3$ m/s). The consistent pair of the figures correspond to directions of unit vector of source along motion direction ($l_1 = 1$, $l_2 = 0$ for Figs. 13, 15, 17), and perpendicular to motion direction ($l_1 = 0$, $l_2 = 1$ for Figs. 14, 16, 18). As it was used above, the curves marked subscript "1" correspond to the results for quasi-shear waves and the curves marked subscript "2" correspond to the results for quasi-longitudinal waves.

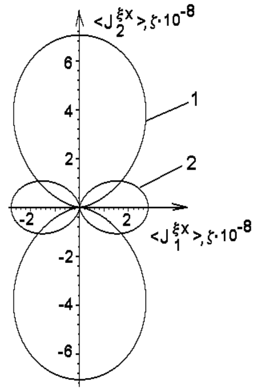


Fig. 13. Energy flux curves,
fused silica, $w_1 = w_2 = 0$, $l_1 = 1$.

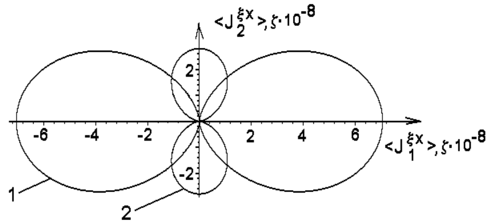


Fig. 14. Energy flux curves,
fused silica, $w_1 = w_2 = 0$, $l_2 = 1$.

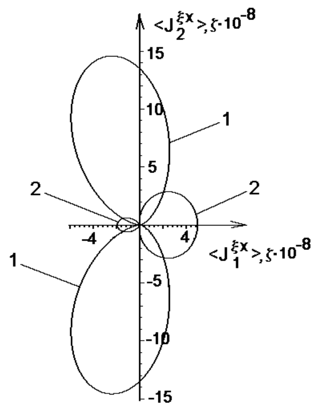


Fig. 15. Energy flux curves,
fused silica, $w_1 = 2600$, $w_2 = 0$, $l_1 = 1$.

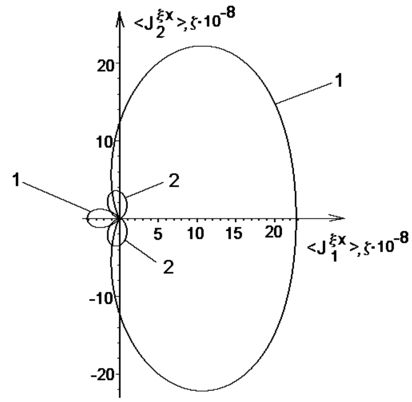


Fig. 16. Energy flux curves,
fused silica, $w_1 = 2600$, $w_2 = 0$, $l_2 = 1$.

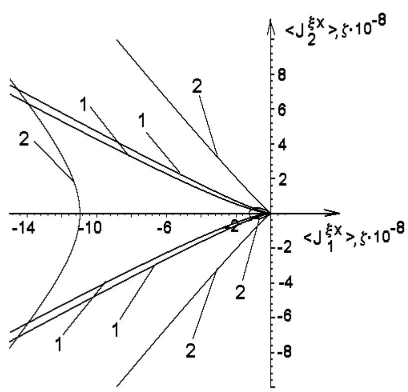


Fig. 17. Energy flux curves,
fused silica, $w_1 = 7500$, $w_2 = 0$, $l_1 = 1$.

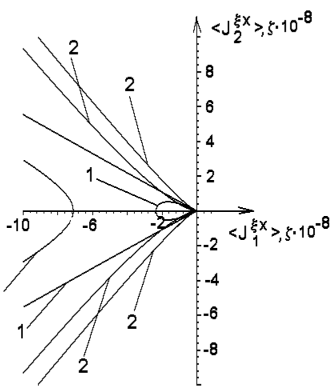


Fig. 18. Energy flux curves,
fused silica, $w_1 = 7500$, $w_2 = 0$, $l_2 = 1$.

From the Figs. 13, 14 it is obvious, that in the problem A for isotropic material fused silica there is the maximum of energy flux for quasi-share wave in direction, which is perpendicular to direction of unit vector of source. All curves of energy source in the problem A are symmetrical to the origin of coordinate system. For the problem B with subseismic motion (Figs. 15, 16) the curves of energy source change significantly and become not symmetrical. For superseismic motion (Figs. 17, 18) the energy flux curves are not limited, because not far from boundaries of wave

propagation (Mach's cones) the curvature of slowness curves tends to 0. As for the case of slowness curves each couple of energy flux curves with superseismic motion corresponds to fast and slow waves.

The figures 19–22 illustrate the pictures of energy flux behavior for anisotropic material (α-quartz). For the Figs. 19, 20 we have the case of problem A with $\mathbf{w} = 0$ (analogically Figs. 7, 8), and for the Figs. 21, 22 we have the case of problem B with superseismic motion (analogically Figs. 9, 10). In this cases we obtain exceptionally complicated behavior of energy flux vectors. (For obviousness the subscript "1", which corresponds to quasi-shear waves is absent in Figs. 19–22.)

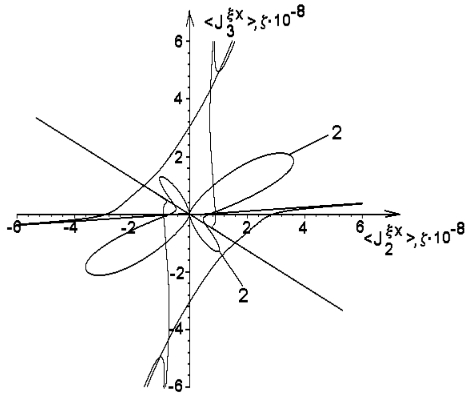


Fig. 19. Energy flux curves, α-quartz, $w_1 = w_2 = 0$, $l_1 = 1$.

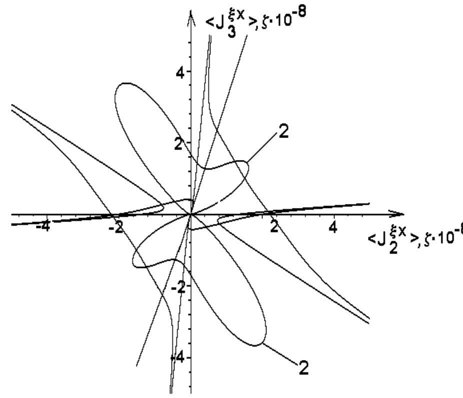


Fig. 20. Energy flux curves, α-quartz, $w_1 = w_2 = 0$, $l_2 = 1$.

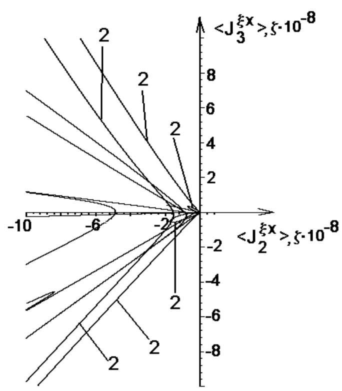


Fig. 21. Energy flux curves, α-quartz, $w_1 = 8000$, $w_2 = 0$, $l_1 = 1$.

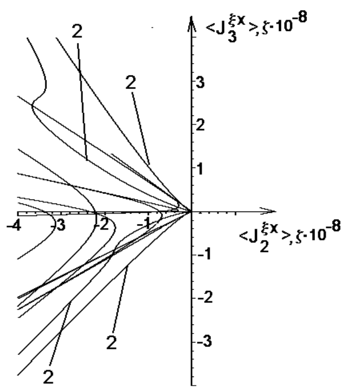


Fig. 22. Energy flux curves, α-quartz, $w_1 = 8000$, $w_2 = 0$, $l_2 = 1$.

In the conclusion we announce similar paper [14], where additional pictures of phase velocity are presented, but the results of energy flux calculation are absent.

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